(see Fig. 1). The equations are

$$a_\xi = -(p/2)\xi[1 - (3\eta/R_0) + (3/2R_0^2)(4\eta^2 - \xi^2) + \ldots] \cong -(p/2)\xi \quad \text{m/sec}^2$$

$$a_\eta = p\eta[1 + (3/4R_0\eta)(\xi^2 - 2\eta^2) + \ldots] \cong p\eta \quad \text{m/sec}^2$$

where $p = 2gR_e^2/R_0^3 = 2.38 \times 10^{-6}$ sec^{-2} for a 96-min orbit. These are the tidal accelerations in the ξ and η directions, respectively.

It is noted that, at ξ or η = 100 km, the second-order term in the equations affect the tidal acceleration by less than 5%. Therefore, the tidal accelerations are close linear functions of ξ and η.

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Planar Librations of an Extensible Dumbbell Satellite

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If a gravitationally stabilized satellite has a "tip mass" connected to it through a long elastic arm, the librational motion about the local vertical will excite longitudinal or "pumping" vibrations of the elastic arm. If the "pumping" motions are damped out by some type of internal (passive) friction, the librational motion will be damped also. In this paper, the equations of planar motion for such a system are derived and then linearized for the case of viscous damping in a circular orbit. The complete solution is given in terms of two nondimensional parameters, which are a measure of spring stiffness and viscous damping, respectively. It is shown that, if the internal friction arises from "material damping" within the spring, there will be relatively little damping of a viscous nature; however, there is a nonlinear time-independent type of hysteretic damping which could be significant. It is shown how the latter type of damping may be analyzed by a technique of "equivalent viscous damping." A configuration of practical interest is examined in a numerical example.

Nomenclature

A = amplitude of spring oscillation
a = parameter defined by Eq. (3.15)
C = viscous damping constant
C_{cr} = $2\bar{m}p_\theta(n^2 - 1)^{1/2}$
D = helical spring coil diameter
D_m = unit damping energy of material
d = wire diameter of helical spring
e = base of natural logarithms
F_d = damping force
G = modulus of elasticity in shear (modulus of rigidity)
g_0 = acceleration of gravity at surface of earth
K = material damping constant
k_s = spring constant, force/unit extension
m_1 = satellite mass
m_2 = tip mass
$\bar{m}$ = $m_1m_2/(m_1 + m_2)$
n = $p_s/p_\theta = (k_s/3\bar{m}\Omega^2)^{1/2}$
N_c = number of coils in helical spring
p_θ = $3^{1/2}\Omega$ = uncoupled librational circular frequency
p_s ≡ $(k_s/\bar{m})^{1/2}$ = uncoupled pumping circular frequency
q_r, q_θ = generalized coordinates
Q_r, Q_θ = generalized forces
r = distance between masses m_1 and m_2
r_0 = value of r when spring is unstrained
R_0 = radius of earth
S = spring force
s, s_{ij} $(i = 1,2)$ = roots of characteristic equation
T = kinetic energy of system
T_c = kinetic energy of mass center
T_1, T_2 = kinetic energies of m_1 and m_2 with respect to mass center
t = time
T_0 = orbital period
U = potential energy stored in spring
ΔU = energy loss/cycle
x = extension of spring from free length
x_{st} = $r_0/(n^2 - 1)$ = static extension of spring in orbit
z = $(x - x_{st})/r_0$
Z = initial value of z
$\alpha_i (i = 1,2)$ = negative of real part of root of characteristic equation
$\beta_i (i = 1,2)$ = imaginary part of root of characteristic equation
γ = $C/\bar{m}p_\theta$
$\delta(\ )$ = virtual displacement in ( )
ζ = $|\mu_1|n^2/\pi\beta_1(n^2 - 1)^{1/2}$
Θ = initial libration amplitude
θ = libration angle
θ_1, θ_2 = constants of integration
θ_m = $\Theta e^{-t/\tau}$
Λ = r_0d/N_cD^2
λ = $\frac{1}{2}\Delta U/U$ = logarithmic decrement
λ_m = logarithmic decrement of material
μ = Z/Θ
μ_1, μ_2 = μ evaluated for s_{11} or s_{21}
ν = $g_0R_0^2$
ρ, ρ_1, ρ_2 = radius from center of earth to mass center of satellite, m_1, m_2, respectively

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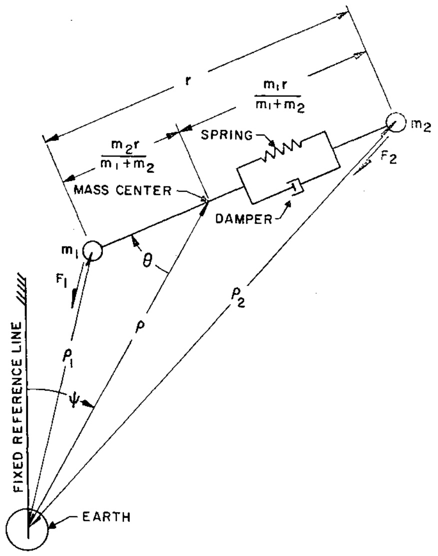

Fig. 1 Satellite system in orbital plane

τ = $1/(3^{1/2}\,\Omega\alpha_1)$ = system time constant
τ_m = shear stress amplitude
$\varphi, \varphi_1, \varphi_2$ = phase angles
ψ = anomaly
Ω = $\dot{\psi}$ = angular speed in circular orbit
ω = circular frequency of extension vibration

1. Introduction

It is well known that an elongated satellite in orbit around the earth will tend to align its "long" principal axis (axis of minimum moment of inertia) along the local vertical. Any small disturbance of this axis from its vertical equilibrium position will cause a vibration that generally is referred to as the *librational* motion. It has been shown repeatedly[1,2] that such a libration, once excited, will continue indefinitely unless some mechanism is present to dissipate energy. Among the various schemes considered to provide damping of the librational motion is one analyzed in some detail by Hall and Smith,[3] who attributed the basic idea to R. E. Roberson. Roberson[4] apparently had discussed similar schemes at an earlier date, although not in sufficient detail for practical design purposes. The same idea was arrived at independently some time afterward at Applied Physics Laboratory of Johns Hopkins University, where the concept was incorporated into the TRAAC satellite.[5] This design consists primarily of two masses interconnected along the axis of minimum inertia by means of an elastic spring and an energy dissipating device, as shown schematically in Fig. 1. A librational motion induces centripetal accelerations that excite longitudinal vibrations of the spring and consequently cause damping of both the spring vibration and satellite libration.

The purpose of this paper is to obtain an estimate of the rate of libration attenuation in practical satellite systems where the dissipation is provided by the so-called "material damping" within the microstructure of the spring material. The satellite is assumed to have an inexorable circular orbit about a spherical earth and to be uninfluenced by any force fields other than gravitational attraction of the earth. It is to be understood that only the "pitch" motions (in orbital plane) of the satellite are under consideration in this paper. Whether or not the simple configuration illustrated in Fig. 1 is capable of damping motions perpendicular to the orbital plane can be determined only by a study of the three-dimensional problem. Such a study is beyond the scope of this paper. Other effects that have not been included in this paper but that might influence the behavior of an actual satellite include bending, twisting, and elastic wave propagation in the spring, time, temperature, and "static stress" effects on the spring material, forced vibrations due to orbit eccentricity, earth-bulge, and other perturbations.

In Ref. 3, the damping is assumed to be linearly viscous ab initio; however, the general equations of motion were not linearized but solved numerically for certain parameter values. The explicit solution of the linearized equations presented herein enables the analysis to proceed without any detailed specialization of initial conditions or system parameters, within limits of the *small deflection* requirements.

The importance of obtaining fairly general solutions, such as those obtained by proper linearization, cannot be overemphasized. For example, in a recent survey of attitude control problems,[6] the authors have concluded, on the basis of Hall and Smith's work, that the system under discussion does not constitute a practical attitude control scheme. This conclusion is based on a very special combination of parameters. In order to do justice to the basic concept, it is necessary to examine a much wider range of parameters. The analysis of this paper makes it possible to make such a study without direct recourse to electronic computers. Finally, the method used in this paper for treating material damping in multidegree of freedom systems, via the concept of equivalent viscous damping, seems to the author to be of value and is capable of being extended to more complex problems.

2. Equations of Motion

The equations of motion for the system will be derived by use of the Lagrangian formulation (Ref. 8, Chap. 3). One generalized coordinate of the problem will be the angle θ between the local vertical and the line joining the two point masses m_1 and m_2 shown in Fig. 1. The second generalized coordinate will be the distance r between the two masses.

A. Kinetic Energy

The kinetic energy T is found easily by noting that the kinetic energy T_c of a particle, of mass $m = m_1 + m_2$, moving with the speed v of the mass center is given by (cf., Fig. 1)

$$T_c = \tfrac{1}{2} mv^2 = (m/2)[\dot{\rho}^2 + (\rho\Omega)^2] \tag{2.1}$$

where $\Omega = \dot{\psi}$ is the angular velocity of the line joining the center of the earth to the satellite mass center, ρ is the radius vector from earth center to the satellite mass center, and ψ is the angle (or anomaly) between a reference line (e.g., line of apsides) and radius vector. The total kinetic energy of the system (with respect to earth-fixed axes) is given (Ref. 8, p. 158) by

$$T = T_c + T_1 + T_2 \tag{2.2}$$

where

$$T_1 = (m_1/2)[(m_2\dot{r}/m)^2 + (m_2 r/m)^2(\dot{\theta} + \Omega)^2] \tag{2.3}$$

and similarly for T_2 with interchange of subscripts 1 and 2. Hence by Eqs. (2.1–2.3)

$$T = (m/2)[\dot{\rho}^2 + (\rho\Omega)^2] + (\bar{m}/2)[(\dot{r})^2 + r^2(\dot{\theta} + \Omega)^2] \tag{2.4}$$

where the "reduced mass" $\bar{m}$ is defined by

$$\bar{m} = m_1 m_2/(m_1 + m_2) = m_1 m_2/m \tag{2.5}$$

B. Potential Energy and Generalized Forces

Within the framework of idealizations chosen for this analysis, the only forces that act on the masses m_1 and m_2 are: 1) gravity force, 2) spring force, and 3) damping force. Of these, the first two are conservative and may be derived from a potential function, whereas the last cannot be so derived.

The potential of the spring forces is shown easily to be equal to the strain energy U stored in the spring, which is a function only of the elongation $r - r_0$, where r_0 designates the free length of the spring.

The potential V_i of a spherical mass m_i in an inverse square force field is given by the well-known expression (Ref. 9, p. 76).

$$V_i = -\nu m_i/\rho_i \qquad (i = 1,2) \quad (2.6)$$

$$\nu = g_0 R_0^2 \quad (2.7)$$

where g_0 represents the acceleration of gravity at the earth's surface and R_0 is the radius of the spherical earth. In order to express ρ_i in terms of generalized coordinates, one notes from Fig. 1 and the law of cosines that

$$\rho_i^2 = \rho^2 + (m_2 r/m)^2 \mp 2\rho(m_2 r/m)\cos\theta \quad (2.8)$$

where the upper sign is used for $i = 1$, the lower sign is used for $i = 2$, and $m \equiv m_1 + m_2$. It proves convenient to express the term $1/\rho_i$ in powers of r/ρ (e.g., by use of the binomial theorem) as follows:

$$\frac{1}{\rho_i} = \frac{1}{\rho}\left[1 - \frac{1}{2}\left(\frac{m_2}{m}\frac{r}{\rho}\right)^2 + \frac{m_2}{m}\left(\frac{r}{\rho}\cos\theta\right) + \frac{3}{2}\left(\frac{m_2}{m}\frac{r}{\rho}\cos\theta\right)^2 + \ldots\right] \quad (2.9)$$

where terms of order $(r/\rho)^3$ have been omitted inside the brackets. To find $1/\rho_2$, merely replace m_2 by $-m_1$ on the right-hand side of Eq. (2.9). Upon substitution of ρ_1 and ρ_2 into Eq. (2.6), one finds for the potential V of the gravity forces the expression

$$V = V_1 + V_2 \doteq -(\nu m/\rho) - (\nu\bar{m}/2\rho^3)r^2(1 + 3\cos 2\theta) \quad (2.10)$$

Although the damping forces that act on each mass are not derivable from a potential function, one notes that the work δW done by these forces, in an infinitesimal virtual displacement δr, is $\delta W = -F_d\delta r$, where F_d is the force exerted by the damper on either mass (defined to be positive when tending to separate the masses). The generalized force Q_r corresponding to the coordinate r is defined (Ref. 9, p. 16) by $Q_r = \delta W/\delta r = -F_d$. Since the force F_d does no work when r is held fixed and θ is varied, it follows that the generalized force Q_θ corresponding to the coordinate θ vanishes.

C. Lagrange's Equations of Motions

Lagrange's equations (Ref. 8, p. 215), are

$$\begin{aligned} &(d/dt)(\partial T/\partial\dot{\theta}) - (\partial T/\partial\theta) + [\partial(V + U)/\partial\theta] = 0 \\ &(d/dt)(\partial T/\partial\dot{r}) - (\partial T/\partial r) + [\partial(V + U)/\partial r) = -F_d \end{aligned} \quad (2.11)$$

where a dot signifies differentiation with respect to time t. Upon substitution of T and V into Eq. (2.11), one finds

$$\begin{aligned} &\ddot{\theta} + 2(\dot{r}/r)(\dot{\theta} + \Omega) + (3\nu/2\rho^3)\sin 2\theta = -\dot{\Omega} \\ &\bar{m}\ddot{r} + F_d + (\partial U/\partial r) - r\bar{m}(\dot{\theta} + \Omega)^2 - (\nu\bar{m}r/2\rho^3)(1 + 3\cos 2\theta) = 0 \end{aligned} \quad (2.12)$$

These equations agree, except for notation, with equivalent equations derived in Ref. 3 for the special case where $m_1 = m_2$.

D. Small Motions about Circular Orbit

Now concentrate on the case of a circular orbit such that ρ = constant, and the angular velocity Ω is found by observing that the "centrifugal force" $m\rho\Omega^2$ is balanced by the gravity force $\nu m/\rho^2$, or

$$\nu/\rho^3 = \Omega^2 \quad (2.13)$$

In addition, attention is restricted to small motions such that $\sin 2\theta \approx 2\theta$, $\cos 2\theta \approx 1$, and the following notation is introduced:

$$r = r_0 + x \quad (2.14)$$

where r_0 is the free length of the spring and $x \ll r_0$. At the same time, assume that the spring force is perfectly elastic and expressed by

$$\partial U/\partial r = k_s(r - r_0) = k_s x \quad (2.15)$$

where k_s is the force required to stretch the spring through unit extension.

If one now assumes that all the terms x/r_0, θ, $\Omega\dot{x}/r_0$, $\dot{\theta}/\Omega$ are negligible in comparison to unity, it follows after substitution of Eqs. (2.14) and (2.15) into Eqs. (2.12) that

$$\ddot{\theta} + 2\Omega(\dot{x}/r_0) + p_\theta^2\theta = 0 \quad (2.16a)$$

$$(\ddot{x}/r_0) + (F_d/r_0\bar{m}) + (n^2 - 1)p_\theta^2(x/r_0) - 2\Omega\dot{\theta} = p_\theta^2 \quad (2.16b)$$

where

$$p_\theta = 3^{1/2}\Omega \qquad p_s = (k_s/\bar{m})^{1/2} \qquad n = p_s/p_\theta \quad (2.17)$$

The parameters p_θ and p_s have the following physical significance: p_θ is the circular frequency of libration for a rigid dumbbell; p_s is the circular frequency for the pumping mode in the absence of gravity, damping, or orbital motion.

3. System with Linear Viscous Damping

If the damping force is proportional to the rate of extension, one may write

$$F_d = C\dot{x} \quad (3.1)$$

where C will be referred to as the damping constant.

In order to place the equations in a more symmetric form, the nondimensional variable z is defined by

$$z = (x - x_{st})/r_0 \quad (3.2)$$

where

$$x_{st}/r_0 \equiv 1/(n^2 - 1) \quad (3.3)$$

As may be seen from Eq. (2.16b), x_{st} represents the deflection of the spring when both $\dot{x}$ and $\dot{\theta}$ are zero. That is, x_{st} is in the nature of a "static" deflection when the system is orbiting without libration. It should be noted from Eq. (3.3) that x_{st} grows beyond bound when $n = 1$. The physical significance of this fact is that the spring cannot support the tension produced by the orbital motion unless $p_s > p_\theta$. Mathematically, this is manifested by an instability of the equations of motion unless $n > 1$.

Upon substitution of Eqs. (3.1) and (3.2) into the equation of motion (2.16), it is found that the governing equations of the system assume the form

$$\ddot{\theta} + 2\Omega\dot{z} + p_\theta^2\theta = 0 \quad (3.4a)$$

$$\ddot{z} + (C/\bar{m})\dot{z} - 2\Omega\dot{\theta} + (n^2 - 1)p_\theta^2 z = 0 \quad (3.4b)$$

In order to solve these equations, the standard trial solution is adopted:

$$\theta = \Theta\exp(sp_\theta t) \qquad z = Z\exp(sp_\theta t) \quad (3.5)$$

Upon substitution of the trial solution into the equations of motion, one finds that s must satisfy the *characteristic equation:*

$$s^4 + \gamma s^3 + (n^2 + \tfrac{4}{3})s^2 + \gamma s + n^2 - 1 = 0 \quad (3.6)$$

where the symbol γ has been used to represent

$$\gamma = C/\bar{m}p_\theta = C/3^{1/2}\bar{m}\Omega \quad (3.7)$$

and the ratio Z/Θ, hereafter specified by μ, must satisfy

$$\mu \equiv Z/\Theta = -3^{1/2}(s^2 + 1)/2s \quad (3.8)$$

When damping is not too great and the system is dynamically stable, the four roots of Eq. (3.8) appear as two pairs of

Table 1 α_1

C/C_{cr}	$n = 2^{1/2}$	2	4	6
0.01	0.002500053	0.001541552	0.0002159679	0.00006298106
0.05	0.01250660	0.007705733	0.001079315	0.0003148277
0.10	0.02505294	0.01539866	0.002155352	0.0006291700
0.15	0.03767954	0.02306525	0.003224861	0.0009425431
0.20	0.05042853	0.03069057	0.004284637	0.001254468
0.25	0.06334449	0.03825748	0.005331552	0.001564475
0.30	0.07647557	0.04574579	0.006362578	0.001872099
0.40	0.1036008	0.0638557	0.008365454	0.002478405

Table 2 α_2

C/C_{cr}	$n = 2^{1/2}$	2	4	6
0.01	0.007499947	0.01577896	0.03851386	0.05909782
0.05	0.03749340	0.07889681	0.1925698	0.2954892
0.10	0.07494706	0.1578064	0.3851430	0.5909788
0.15	0.1123204	0.2367424	0.5777226	0.8864694
0.20	0.1495715	0.3157196	0.7703120	1.181961
0.25	0.1866555	0.3947552	0.9629143	1.477455
0.30	0.2235244	0.4738694	1.155532	1.772952
0.40	0.2963992	0.6324347	1.540828	2.363953

Table 3 β_1

C/C_{cr}	$n = 2^{1/2}$	2	4	6
0.01	0.5773530	0.7994324	0.9557892	0.9809714
0.05	0.5774180	0.7996270	0.9558132	0.9809763
0.10	0.5776223	0.8002363	0.9558881	0.9809916
0.15	0.5779662	0.8012559	0.9560125	0.9810170
0.20	0.5784551	0.8026917	0.9561858	0.9810525
0.25	0.5790969	0.8045516	0.9564069	0.9810980
0.30	0.5799023	0.8068455	0.9566748	0.9811533
0.40	0.5820638	0.8127804	0.9573450	0.9812928

Table 4 β_2

C/C_{cr}	$n = 2^{1/2}$	2	4	6
0.01	1.732010	2.166539	4.051948	6.030549
0.05	1.731035	2.164536	4.047448	6.023565
0.10	1.727984	2.158262	4.033355	6.001687
0.15	1.722882	2.147767	4.009756	5.965045
0.20	1.715703	2.132988	3.976486	5.913367
0.25	1.706409	2.113841	3.933300	5.846251
0.30	1.694951	2.090210	3.879871	5.763162
0.40	1.665269	2.028877	3.740460	5.546051

conjugate complex numbers with negative real parts that may be expressed in the form

$$\left.\begin{matrix} s_{11} \\ s_{12} \end{matrix}\right\} = -\alpha_1 \pm i\beta_1 \qquad \left.\begin{matrix} s_{21} \\ s_{22} \end{matrix}\right\} = -\alpha_2 \pm i\beta_2 \tag{3.9}$$

where α_1 and α_2 are real positive numbers and $i = (-1)^{1/2}$. Under these circumstances, the complete solution may be written in the real form

$$\begin{aligned} \theta &= \Theta_1 e^{-\alpha_1 p_\theta t} \cos(p_\theta \beta_1 t + \varphi_1) + \Theta_2 e^{-\alpha_2 p_\theta t} \times \cos(p_\theta \beta_2 t + \varphi_2) \\ z &= |\mu_1| \Theta_1 e^{-\alpha_1 p_\theta t} \cos(p_\theta \beta_1 t + \varphi_1 + \arg\mu_1) + |\mu_2| \Theta_2 e^{-\alpha_2 p_\theta t} \cos(p_\theta \beta_2 t + \varphi_2 + \arg\mu_2) \end{aligned} \tag{3.10}$$

In these equations, Θ_1, Θ_2, φ_1, φ_2, represent real constants determined by the initial conditions; $\mu_1 = \mu(s_{11})$, $\mu_2 = \mu(s_{21})$. where $\mu(s)$ is the complex number defined by Eq. (3.8). The notations $|\mu|$ and $\arg\mu$ denote the absolute value of μ and argument (angle) of the complex number μ, respectively.

In order to make further progress, it is desirable to investigate some numerical cases at this point. The roots of the characteristic Eq. (3.8) depend upon the two dimensionless parameters n^2 and γ. It already has been decided that n^2 should be greater than unity for stability, and the following cases will be investigated numerically: $n^2 = 1,2,4,16,36,64, 100$. In order to choose sensible values of $\gamma = C/\bar{m}p_\theta$, it is noted that, if the librational motion is suppressed, i.e., if $\dot{\theta} = 0$, Eq. (3.4b) describes a viscously damped harmonic oscillator whose critical damping constant C_{cr} is defined (Ref. 8, p. 35) by

$$C_{cr} = 2\bar{m}p_\theta(n^2 - 1)^{1/2} \tag{3.11}$$

Hence, from Eq. (3.7),

$$\gamma = C/\bar{m}p_\theta = (C/C_{cr})(C_{cr}/\bar{m}p_\theta) = 2(n^2 - 1)^{1/2}(C/C_{cr}) \tag{3.12}$$

Tables 1–4 show typical values of α_i, β_i, corresponding via Eqs. (3.9) to the roots of the characteristic evuation, over a range of interest of n and C/C_{cr}.

The following points of interest may be noted from a study of the tables:

1) For n greater than 2, β_1 is very close to unity and β_2 is very close to n, at least for values of C/C_{cr} less than about $\frac{1}{2}$.

Hence, it is seen that for $n > 2$ one of the two component oscillations occurs at essentially the uncoupled librational frequency $p_\theta = 3^{1/2}\Omega$; the other component oscillation occurs essentially at the uncoupled pumping frequency $np_\theta = p_s = (k_s/\bar{m})^{1/2}$ [cf., Eq. (2.17)].

2) For n greater than 2, α_2 is considerably greater than α_1. This means that the component of θ oscillating at "pumping" frequency is damped out considerably faster than the component oscillating at "librational" frequency.

Therefore, regardless of the initial conditions, the system settles down after a short time to a motion where both θ and z oscillate at essentially "librational frequency" p_θ, with damping rate determined by α_1. Thus, after a reasonable length of time, the motion can be represented [cf., Eq. (3.10)] by

$$\begin{aligned} \theta &\approx \Theta e^{-t/\tau} \cos(\beta_1 p_\theta t + \Phi) \\ z &\approx \mu|\Theta| e^{-t/\tau} \cos(\beta_1 p_\theta t + \Phi + \arg\mu) \end{aligned} \tag{3.13}$$

where Θ, Φ are constants determined by initial conditions, and the "time constant" τ of the system is the time required for the amplitudes to diminish by a factor of $1/e = 0.368$ and is defined by

$$\tau = 1/3^{1/2}\Omega\alpha_1 = T_0/10.9\alpha_1 \tag{3.14}$$

The expression $T_0 = 2\pi/\Omega$ represents the orbital period. Figure 2 shows how the ratio T_0/τ varies with the "damping ratio" C/C_{cr} for various values of the "frequency ratio" $n = p_s/p_\theta$. The ratio τ/T_0 is precisely the number of orbits which must be completed before the amplitude decays by 63%.

A close examination of Table 1 shows that for light damping α_1 is practically proportional to C/C_{cr} for any fixed value of n; that is,

$$\alpha_1 \approx a\ (C/C_{cr}) \tag{3.15}$$

where a depends upon n. Figure 3 shows how a should be chosen to make Eq. (3.15) exact for $C/C_{cr} = 0.2$ and never cause an error exceeding 1% for smaller values of C/C_{cr}.

The numerical value of C/C_{cr} depends upon the materials and construction of the actual system. In any practical system, at least a part of the damping must come from hysteretic loss in the material of the spring, that is, from "material damping." The next section will be devoted to some considerations of material damping.

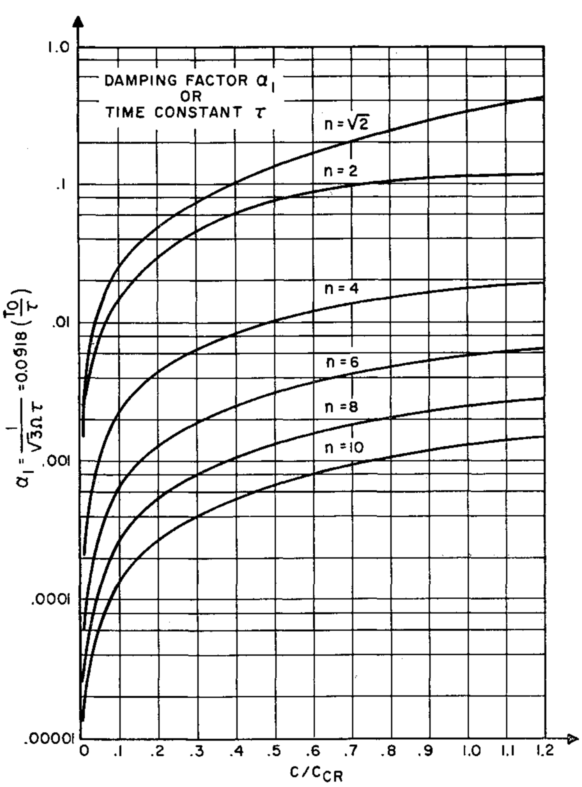

Fig. 2 Damping factors

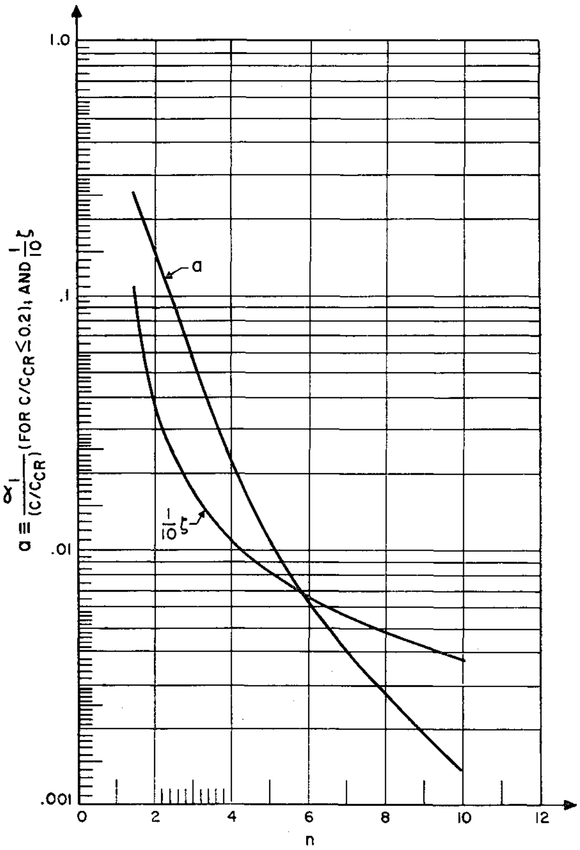

Fig. 3 Parameter *a* vs *n*

4. Damping in Real Materials

Some of the many factors that influence the phenomenon of energy dissipation in materials are discussed in the recent review of Lazan.[7] It is most convenient for present purposes to divide these factors into two major categories that will be designated as time (or frequency) dependent effects and time (or frequency) independent effects. Time dependency is characteristic of viscosity effects; the microscopic origin of such effects in solid materials is discussed by Zener.[12] From a macroscopic point of view, it is sufficient to know the damping constant C for a given spring or the logarithmic decrement λ, defined in general (for all types of damping) by

$$\lambda \equiv \tfrac{1}{2}(\Delta U/U) \equiv \Delta U/k_s A_s^2 \tag{4.1}$$

where

ΔU = energy dissipated/cycle
k_s = spring constant, force/unit extension
A_s = amplitude of vibration

A. Frequency-Dependent Damping

In the case of a linear damper, with damping constant C, one may show that $\Delta U = \pi C A^2 \omega$, where ω is the circular frequency of vibration. Therefore, Eq. (4.1) may be written in the form

$$\lambda = \pi C\omega/k_s \tag{4.2}$$

It is significant that, for a viscous material, the logarithmic decrement is independent of amplitude but dependent on frequency. For this reason, such damping is frequently called

Table 5 Approximate time constants ($\lambda_m/\pi = 0.002$; $\Omega = 86.4$ day^{-1}; $\beta_1 \approx 1$)

n	$2^{1/2}$	2	4	6	8	10
τ days	13.2	18.9	75.5	175	307	495

"amplitude independent" damping.[7] It will be seen below that materials that exhibit time-independent damping generally have amplitude-dependent logarithmic decrements. Since the present formulation of the general problem requires knowledge of the damping ratio C/C_{cr}, one must express this ratio in terms of the loss factor λ_m, where the subscript m signifies that λ is a material property rather than a system property. The required relation is

$$\left(\frac{C}{C_{cr}}\right) = \frac{\lambda_m}{\pi}\frac{k_s}{\omega C_{cr}} = \frac{\lambda_m}{2\pi}\left(\frac{p_s}{\omega}\right)\frac{n}{(n^2-1)^{1/2}} \qquad (4.3)$$

where use has been made of the definition of C_{cr} given by Eq. (3.11) and the fact that $(p_s/p_\theta) \equiv n$. If one recalls that the oscillation frequency after a reasonable length of time is given by $\omega = \beta_1 p_\theta = \beta_1 3^{1/2}\Omega$, Eq. (4.3) may be written as

$$C/C_{cr} = (\lambda_m/2\pi\beta_1)[n^2/(n^2-1)^{1/2}] \qquad (4.4)$$

Upon substituting Eqs. (4.4) and (3.15) into Eq. (3.14), one finds the time constant

$$\tau \approx \frac{2\beta_1(n^2-1)^{1/2}}{3^{1/2}\Omega a n^2(\lambda_m/\pi)} < \frac{2(n-1)^{1/2}\pi}{3^{1/2}\Omega a n^2\lambda_m} \qquad (4.5)$$

where the inequality follows from $\beta_1 < 1$ (see Table 3). However, since β_1 is practically unity for $n > 2$, the error is not too great in assuming $\beta_1 \approx 1$.

Assume for the moment that the damping is primarily viscoelastic, and try to find the order of magnitude of λ_m based on this assumption. Since viscoelastic loss factors are quite frequency-dependent, it is necessary to use experimental data obtained in the very low frequency range (corresponding to near earth orbits) of the order of 10^{-4} cps. Although the author knows of no experiments performed at such a low frequency, perhaps it may be instructive to examine experimental results corresponding to somewhat higher frequencies in order to obtain a feeling of numerical magnitudes.

Experiments of Bennewitz and Rötger[13] are reported by Zener (Ref. 12, p. 55) for a number of different materials in the frequency range of 1 to 1000 cps. The maximum logarithmic decrements are of the order of $\lambda_m = 0.002\pi$. It therefore is not entirely unreasonable to use this number in the low frequency range of 10^{-3} or 10^{-4} cps. Using this value for λ_m together with $\Omega = 86.4$ rad/day (corresponding to an orbital altitude of 500 naut miles) and the observation that β_1 is approximately unity, one can compute close upper bounds on the time constant from Eq. (4.5) and Fig. 3, with the results shown in Table 5.

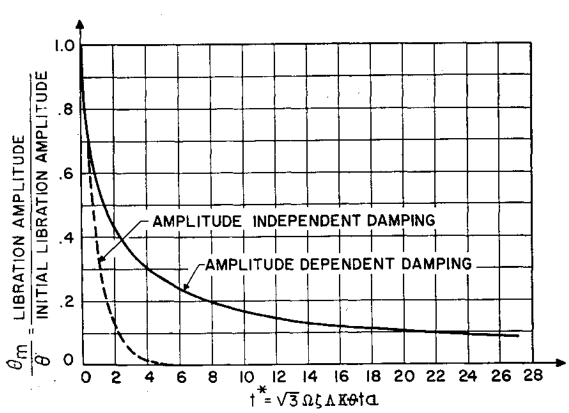

Fig. 4 Libration amplitude vs t^*

Table 6 Functions $|\mu_1|$ and ζ

n	C/C_{cr}	$\lvert\mu_1\rvert$	Mean $\lvert\mu_1\rvert$	$\zeta(n)$
$2^{1/2}$	0.01	0.99998	0.998	1.100
	0.20	0.99615		
2	0.01	0.39097	0.388	0.358
	0.20	0.38375		
4	0.01	0.078344	0.0780	0.107
	0.20	0.077637		

It thus appears that even for very soft springs (low n) the typical available values of λ_m (of order 0.002π) correspond to fairly long time constants. In other words, if one relies upon the viscoelastic damping inherent in most spring materials, one can expect quite small damping rates.

B. Frequency-Independent Damping

It then appears that, if a lower time constant is required, it is necessary to rely upon some mechanical phenomenon other than inherent viscoelastic material damping.

One such phenomenon is time-independent hysteresis. Actually, this term includes a host of submicroscopic phenomena described in Refs. 7, 11, and 15. Attention will be restricted to macroscopic behavior such as may be observed for a closewound helical spring. It may be shown[14] that, if such a spring is made of a material with time-independent damping, for low stresses, the logarithmic decrement λ_m is approximately a linear function of the stress amplitude τ_m, i.e.,

$$\lambda_m/2\pi = K(\tau_m/G) \qquad (4.6)$$

where G is the modulus of elasticity in shear and K is an experimentally determined material constant. Using this expression in Eq. (4.4), one finds that the equivalent viscous damping ratio C/C_{cr} is given by†

$$(C/C_{cr})_{eq} = K\tau_m n^2/G\beta_1(n^2-1)^{1/2} \qquad (4.7)$$

It has been assumed up to now that the shear stress undergoes complete reversal between values of $\pm\tau_m$. If, however, the shear stress oscillates about a "static" stress level τ_{m0} such that its extreme values are $\tau_{m0} \pm \tau_m$, there may be marked differences in the damping losses in the two cases (particularly for magnetostrictive damping). Cochardt[15] has observed that the presence of a static stress usually depresses the damping, but cases are known where a static stress increases the damping.

5. Equivalent Viscous Damping

In order to use the equivalent viscous damping ratio defined by Eq. (4.7), it is desirable to express the shear stress amplitude τ_m in terms of the "librational" amplitude Θ. To do this, one notes from Eq. (3.13) that when the system librates with amplitude θ_m defined by

$$\theta_m \equiv \Theta e^{-\alpha_1 3^{1/2}\Omega t} \qquad (5.1)$$

the variable z oscillates with amplitude $z_m = |\mu_1|\theta_m$. The physical extension of the spring from its free length has been defined by the coordinate x; therefore, the amplitude of oscillation A about the "static" deflection defined by Eq. (3.3) is given by

$$A = x_m - x_{st} = r_0[(x_m - x_{st})/r_0] = r_0 z_m \qquad (5.2)$$

† For steady-state forced oscillations where τ_m is constant, the equivalent of Eq. (4.7) may be found in the standard literature (e.g., Ref. 10, p. 92). For free vibration of a nonlinearly damped system, Eq. (4.7) is equivalent to the result found by the method of Kryloff and Bogoliuboff (Ref. 16, p. 59).

where use has been made of the definition of z given by Eq. (3.2), and x_m signifies the maximum value of x. Since $z_m = |\mu_1|\theta_m$, one may write

$$A = r_0|\mu_1|\theta_m \quad (5.3)$$

for the extension amplitude of the spring from its position of static equilibrium while in orbit.

For a coil spring with N_c coils of wire of diameter d and modulus of rigidity G, wound in a close-coiled helix of diameter D, the maximum shear stress τ_m corresponding to an extension A is given[17] by

$$\tau_m = GdA/\pi N_c D^2 \quad (5.4)$$

Substitution of Eqs. (5.4) and (5.3) into Eq. (4.7) results in the following expression for the equivalent viscous damping ratio:

$$\left(\frac{C}{C_{cr}}\right)_{eq} = \left(\frac{1}{\pi\beta_1}\frac{|\mu_1|\, n^2}{(n^2-1)^{1/2}}\right) K \left(\frac{dr_0}{N_c D^2}\right)\theta_m \quad (5.5)$$

Equation (5.5) really defines C/C_{cr} implicitly, since β_1 and the complex number μ_1, defined by Eq. (3.8), are dependent upon the root s_1 of the characteristic equation, which in turn depends upon both n and C/C_{cr}. However, now it will be shown that, for light damping, both β_1 and $|\mu_1|$ are essentially independent of C/C_{cr} and hence a function of n only.

From Eq. (3.8), one sees that

$$|\mu_1| = \frac{3^{1/2}}{2}\left|\frac{s_1^2+1}{s_1}\right| = \frac{3^{1/2}}{2}\left[\frac{(1+\alpha_1^2-\beta_1^2)^2+4\alpha_1^2\beta_1^2}{\alpha_1^2+\beta_1^2}\right]^{1/2} \quad (5.6)$$

where α_1 and β_1 are given in Tables 1 and 3.

Table 6 shows the numerical value of $|\mu_1|$ for $C/C_{cr} = 0.01$, $C/C_{cr} = 0.20$, and a variety of values of n. The difference in $|\mu_1|$ for the two different values of C/C_{cr} clearly is negligible in all cases. A more detailed computaton shows that $|\mu_1|$ is very insensitive to C/C_{cr} and may be replaced by its mean value in the range $0.01 < C/C_{cr} < 0.20$. A glance at Table 3 shows that β_1 is also very insensitive to C/C_{cr} in this range; hence, the bracket in Eq. (5.5) is a function of n only, which will be designated by $\zeta(n)$:

$$\zeta(n) = |\mu_1|n^2/\pi\beta_1(n^2-1)^{1/2} \quad (5.7)$$

The function $\zeta(n)$ is tabulated in Table 6, and $\zeta/10$ is shown in Fig. 3. The equivalent viscous damping ratio now takes the simple form

$$(C/C_{cr})_{eq} = \zeta K\Lambda\theta_m \quad (5.8)$$

where Λ is a "geometric" or "form" factor defined by

$$\Lambda = r_0 d/N_c D^2 \quad (5.9)$$

For the case of light damping, where, as was seen in Sec. 3, the damping factor $\alpha_1 \approx a(C/C_{cr})$, the libration amplitude may be written as

$$\theta_m = \Theta \exp[-a(C/C_{cr})_{eq}3^{1/2}\Omega t] \quad (5.10)$$

Upon substitution of $(C/C_{cr})_{eq}$ from Eq. (5.8), one finds that Eq. (5.10) leads to the result

$$\theta_m/\Theta = \exp(-a\zeta K\Lambda\, 3^{1/2}\Omega\theta_m t) \quad (5.11)$$

or

$$-\frac{\ln(\theta_m/\Theta)}{(\theta_m/\Theta)} = t^* \quad (5.12)$$

where t^* is defined by

$$t^* = 3^{1/2}\Omega a\zeta\Lambda K\Theta t = 3^{1/2}2\,\pi a\zeta K\Lambda\Theta(t/T_0) \quad (5.13)$$

Equation (5.12) is illustrated by the curve labeled "amplitude-dependent damping" in Fig. 4. For comparison purposes, the exponential curve that results when the damping proceeds at its initial rate [i.e., when θ_m is replaced by Θ on the right-hand side of Eq. (5.11)] is shown by the broken line in Fig. 4, labeled "amplitude-independent damping."

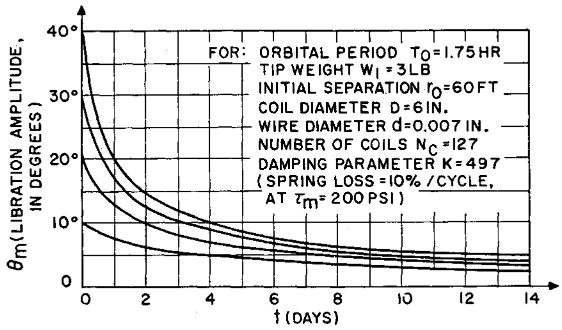

Fig. 5 Libration decay for various initial amplitudes

It perhaps is worth mentioning that, although the present system has two degrees of freedom, the method of solution used here is, in a sense, a variant of the method of Kryloff and Bogoliuboff[16] for systems with a single degree of freedom. One is able to apply the method because one of the two degrees of freedom in the present problem is damped very rapidly. The remaining degree of freedom is then described adequately by one of the principal modes of the undamped system.

6. Numerical Example

The following numerical data will be assumed in order to calculate a typical set of decay times:

Orbit altitude = 500 naut miles (ρ = 4534 statute miles)
Tip mass m_2 weighs 3 lb at earth's surface
For the spring, free length r_0 = 60 ft, number of coils N_c = 127, coil diam D = 6 in., wire diam d = 0.007 in., shear modulus $G = 12 \times 10^6$ psi

From Eqs. (2.13) and (2.7), one finds $\Omega = 1.00 \times 10^{-3}$ rad/sec and $T_0 = 2\pi/\Omega = 1.75$ hr. From Ref. 17, $k_s = 14.2 \times 10^{-3}$ lb/in. From Eq. (2.5), $\bar{m} = m_2/(1 + m_2/m_1) \approx m_2$, since $m_1 \gg m_2$. Therefore, Eq. (2.71) gives $n = 2.46$. From Eq. (5.9) $\Lambda = 1.10 \times 10^{-3}$, and from Fig. 3 one finds $a = 0.092$ and $\zeta = 0.250$.

With these numerical values, one finds from Eq. (5.13)

$$t^* = 0.00286K\theta(t/T_0) \quad (6.1)$$

The factor K must come from experiments performed at frequencies of the same order as that which prevails in space ($\Omega \approx 10^{-3}$ sec^{-1}); otherwise creep, relaxation, and other time-dependent effects might mask the time-independent effects one seeks to measure. It is also desirable that the experiments be performed in the presence of a static stress τ_{m0} corresponding to the static extension x_{st} that will occur in orbit.

The author has been informed by R. Fischell, of Applied Physics Laboratory, that energy losses of 10%/cycle have been observed when a cadmium plated wire of 0.007 in. diam was subjected to torsional oscillations with a stress amplitude of τ_m = 200 psi and a period of roughly 1 hr. This corresponds to a logarithmic decrement of $\lambda_m = \frac{1}{2}10\% = 0.05$. Although no static stress was present in these tests, these data will be used to calculate K. If one assumes that the stress level is sufficiently low for Eq. (4.6) to be valid, one finds that

$$K = \lambda_m G/2\pi\tau_m = 497 \quad (6.2)$$

Substituting this value of K into Eq. (6.2) results in the following expression for the nondimensional time t^*:

$$t^* = 0.142\Theta(t/T_0) = 1.95\Theta t \quad \text{days} \qquad (6.3)$$

Equation (6.3) may be used in conjunction with Fig. 4 to find the time of decay for any given value of Θ. Figure 5 shows the attenuation time curves for various "initial" values of libration. It must be borne in mind that Θ is not, strictly speaking, the initial libration amplitude but may be thought of as the libration amplitude at that time where the "pumping mode" has essentially damped out.

7. Conclusion

The equations of motion for a "pumping," librating satellite have been derived for two-dimensional (pitch) motion along a specified plane orbit. For sufficiently small departures from the local vertical, the equations have been linearized. It has been shown that the motion consists of a "librational mode" and a "pumping mode," the latter of which dies out considerably faster than does the former. Thus, after a sufficient length of time, the system librates and pumps at essentially the uncoupled librational frequency, $3^{1/2}$ × orbit frequency. The rate of attenuation depends upon two nondimensional parameters: n, which is a measure of spring stiffness, and C/C_{cr}, a measure of viscosity. The system time constant τ (time for attenuation of 63%) is given in terms of these two parameters by Fig. 2.

It appears unlikely that real materials can provide a sufficient amount of viscous damping at the very low frequencies involved. However, it has been shown that a type of frequency-insensitive damping, referred to as time-independent hysteresis, possibly may provide adequate damping. However, in order to analyze the effects of such an inherently nonlinear phenomenon, it has been necessary to use the concept of equivalent viscous damping. Using this concept, one easily may find the time required to reduce a small (say less than 30°) librational amplitude θ_m to any presrcibed fraction of its initial value Θ. The ratio θ_m/Θ is given in Fig. 4 in terms of the nondimensional time t^*. The decay rates found in a numerical example are not unreasonable for certain satellite applications.

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